The Geometry of Sea Shells

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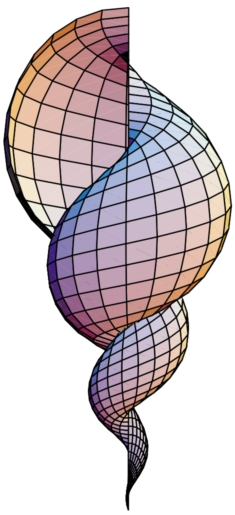
1 The beauty of sea shells

Turritella nivea

A natural description of shell shapes may be given in terms of a **generating spiral** and the shape of the opening or **generating curve**.

For example, consider the surface generated by rotating an expanding semi-circle up the z-axis:

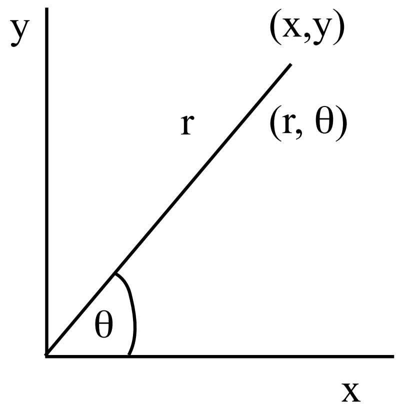


Other mathematical shell surfaces can be generated by rotating more realistic shell openings around helico-spirals.

To learn about such mathematical shapes we need to know more about circles, spirals and parametric descriptions of surfaces.

2 Planar spirals

Points in the plane may be specified with a pair of numbers, such as those of the **Cartesian** coordinate system. Alternatively one may use the planar polar coordinate system

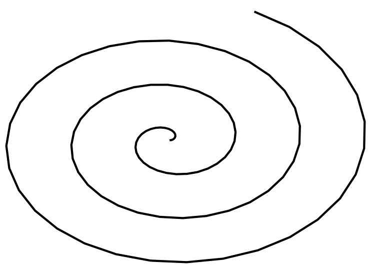


$$x = r\cos\theta$$
$$y = r\sin\theta$$

A one-armed spiral is described by

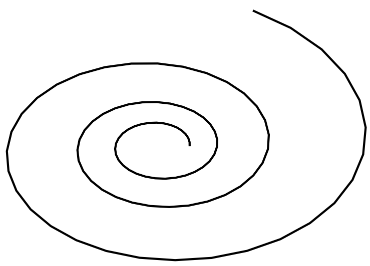
$$\boxed{\theta = f(r)}$$

Examples include the **Archimedean spiral**



$\theta = ar,\ r = \theta/a$

and the **logarithmic spiral**



$\theta = a\ln r,\ r = \exp(\theta/a)$

It is simple to show that the distance between points on an Archimedean spiral for a fixed angle is always

$$\frac{2\pi}{a}$$

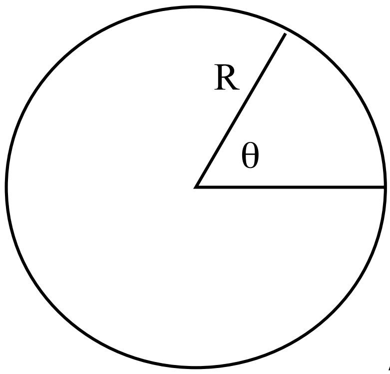
[Q1] What is the answer for a logarithmic spiral?

3 Parametric equation of a circle

A circle of radius R may be described in terms of a single parameter $\theta \in [0, 2\pi)$ as

$$x = R\cos\theta$$
$$y = R\sin\theta$$

If we let θ range over $[0, 2\pi)$ then we generate a **circle**. If θ ranges over $[0, \pi]$ then we generate a **semi-circle**.



A circle of radius R.

[Q2] What does the following describe?

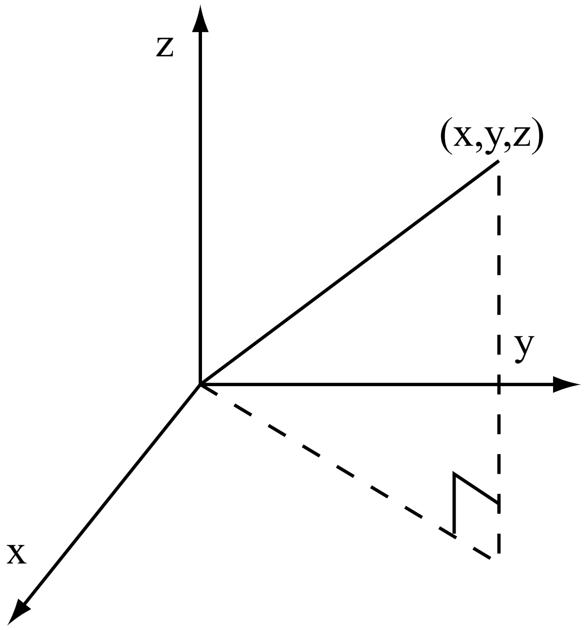
$$x = R\cos\theta$$
$$y = 2R\sin\theta$$

[Q3] How about

$$x = R\sin\theta$$
$$y = R\sin 2\theta$$

4 3D spirals

A point in 3D may be described using the 3D Cartesion coordinate system. In Cartesian coordinates a point is specified with the triple (x, y, z):



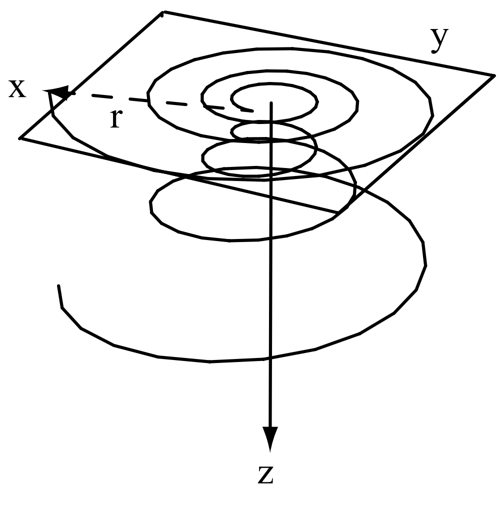
[Q4] Can you work out the the distance from the origin to a point (x, y, z)?
A helico-spiral may be described with the single parameter ϕ if we write

$$r = f_r(\theta)$$
$$z = f_z(\theta)$$

For an 3D Archimedean spiral we may take

$$r = a\theta$$
$$z = b\theta$$

[Q5] How could you make a 3D logarithmic spiral, like the one below?

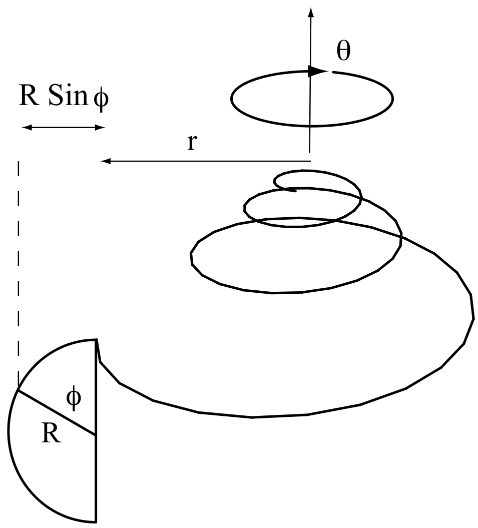


The equation of a helico-spiral has been expressed in terms of a **single** parameter θ.

5 Generation of a simple sea shell

Formed by the rotation of a generating curve along a helico-spiral.

As an example let us consider the generating curve to be a semi-circle and the helico-spiral to be a 3D Archimedean spiral:



To label a point on the surface we need specify how far along the spiral we are (using θ) and how far round the semi-circle we are (using ϕ). This is easily calculated by letting

$$r \rightarrow r + R\sin\phi \qquad z \rightarrow z + R(1 - \cos\phi)$$

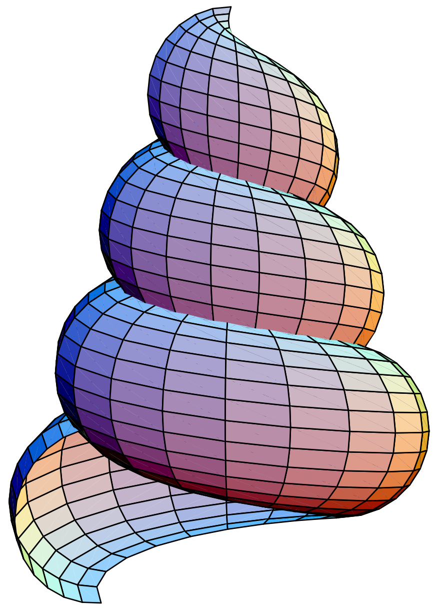
In terms of the 3D Cartesian system the coordinates of the shell are given by

$$(x, y, z) = (r\cos\theta, r\sin\theta, z)$$

Hence, the surface of the shell is completely specified in terms of **two** parameters, θ and $\phi \in [0, \pi)$:

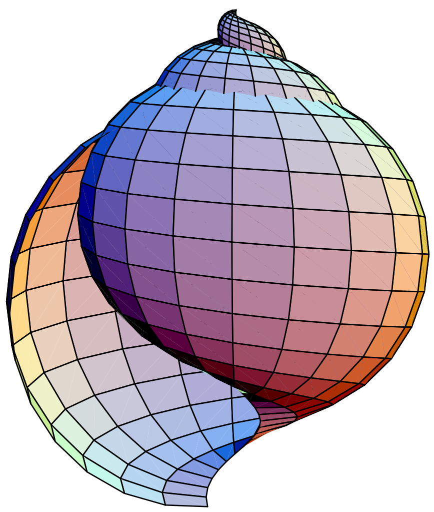
$$x(\theta, \phi) = (a\theta + R\sin\phi)\cos\theta$$
$$y(\theta, \phi) = (a\theta + R\sin\phi)\sin\theta$$
$$z(\theta, \phi) = b\theta + R(1 - \cos\phi)$$

Here is an example of the shell surface just described.

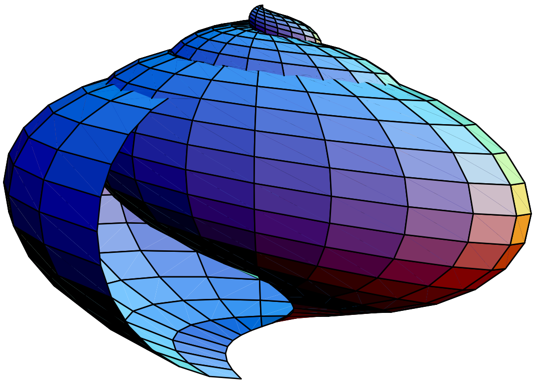


To make more interesting shapes we could use different helico-spirals, make the radius of the semi-circle depend upon θ and ϕ ($R = R(\theta, \phi)$) or even change the shape of the generating curve completely.

Heres what happens if we let the semi-circle expand as it is rotated ($R(\theta) = c(1 + d\theta)$):

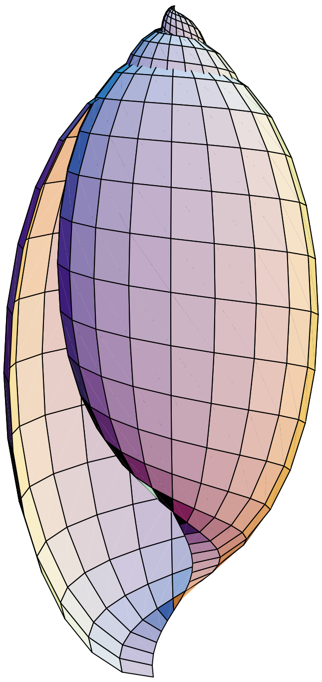


Heres one with more of an ellipsoidal shape.

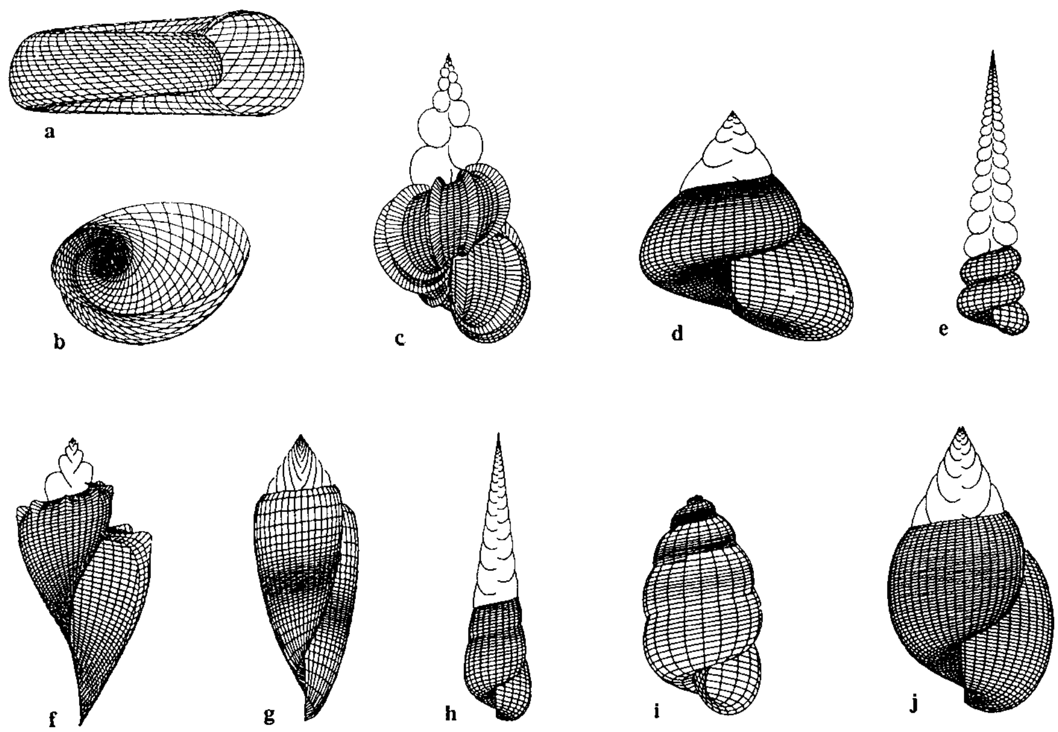


[Q6] How is it made?

[Q7] Can you work out how to generate a shell with an elliptical opening?



Below you will find some examples of shells generated with a logarithmic helico-spiral and various choices of the generating curve.



(a) Planorbis (b) Haliotis (c) Epitonium (d) Oxystele (e) Turritella (f) Lyria (g) Conus (h) Terebra (i) Gulella (j) Achatina.

6 Shell patterns

Have a look at the book **The Algorithmic Beauty of Sea Shells** by Hans Meinhardt, Springer1998.

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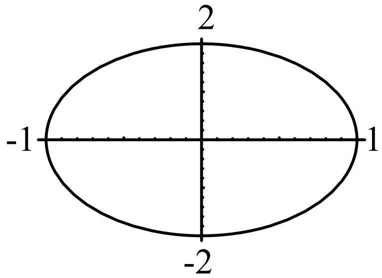
Answers

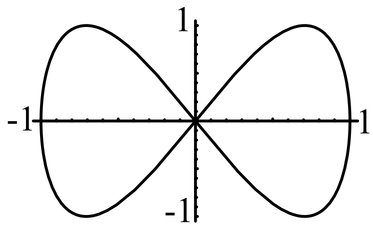
[Q1]

$$\exp\left([\theta + 2n\pi]/a\right)\left[\exp\left(2\pi/a\right) - 1\right], \qquad n \in \mathbb{Z}$$

[Q2] An ellipse



[Q3]



[Q4]

$$\sqrt{x^2 + y^2 + z^2}$$

[Q5]

$$r = \exp(\phi/a)$$
$$z = \exp(\phi/b)$$

[Q6]

$$(x, y, z) = (r\cos\theta, 2r\sin\theta, z)$$

[Q7]

$$r \to r + R\sin\phi \qquad z \to z + 2R(1 - \cos\phi)$$

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